From discrete to continuous dynamics and back: How large is 1?

Hubert Krivine[(a)], Annick Lesne[(b)], Jacques Treiner[(a)*]

(a) LPTMS, Bâtiment 100, Université Paris-Sud, F-91405 Orsay and

(b) LPTL, Case 121, Université Pierre et Marie Curie, 4 place Jussieu, F-75252 Paris.

Discrete autonomous dynamical systems in dimension 1 can exhibit chaotic behavior, whereas the corresponding continuous evolution equations rule it out, and cannot even possess a nontrivial periodic solution. Therefore the passage from discrete to continuous equations (and conversely) is all but harmless. We address this issue and evidence some caveats on the paradigmatic Verhulst logistic equation, investigating in particular the status and influence of the actual size of the unit time step in discrete modelings, rooted in well-known numerical analysis.

I. THE DISCRETE-TIME LOGISTIC EVOLUTION

The logistic map $f_a(x) = ax(1-x)$ giving the celebrated recursion relation on the interval $[0,1]$

$$x_{n+1} = ax_n(1-x_n) = f_a(x_n) \qquad x_0 \in [0,1] \quad a \in]1,4] \tag{1}$$

is one of the simplest example of discrete autonomous evolution leading to chaos. This nonlinear equation was introduced by Verhulst (a Belgian mathematician) in 1838 [1] to take into account that a, the Malthus coefficient characterizing the growth of the population

$$X_{n+1} = aX_n,$$

has to decrease when X_n increases, due to resources limitation. The simplest way was to replace the constant rate a by a linear dependence in X_n, matching the rate a at vanishing population, namely $a(1 - X_n/M)$; the parameter M is then interpreted as being the maximum acceptable population, currently known as the "carrying capacity" of the environment. Equation (1) is recovered through the change of variable $x_n = X_n/M$. A very rich variety of dynamic behaviors is generated by Eq. (1), whose temporal structure is governed by the values of the control parameter a. Since the seminal reference [2], several studies of the asymptotic dynamics of (1) have been published, among which some very pedagogical ones are [3] and [4]. Let us only recall the most significant properties.

* Email addresses for correspondence: krivine@ipno.in2p3, lesne@lptl.jussieu.fr, treiner@ccr.jussieu.fr

For a given, such that $1 < a < a_1 = 3$, the fixed point $x_a^* = 1 - 1/a$ is stable, globally attractive, therefore $x_n \to x_a^*$ as $n \to \infty$, irrespectively of the initial condition x_0 provided it belongs to its basin of attraction $]0,1[$. In $a_1 = 3$, a cycle of period 2 appears through a pitchfork bifurcation. Also called period-doubling bifurcation since it is associated with the destabilization of a fixed point x_a^* into a 2-cycle (or the destabilization of a 2^n-cycle into a 2^{n+1}-cycle when it involves $f_a^{2^n}$ instead of f_a), this generic bifurcation is characterized by the relation $f'_{a_1}(x_{a_1}^*) \equiv \partial_x f(a_1, x_{a_1}^*) = -1$ and the generic condition $\partial_{ax} f(a_1, x_{a_1}^*) \neq 0$ (denoting here the a-dependence on the same footing for the sake of clarity) [5]. The 2-cycle emerging in a_1 remains stable and globally attractive in $]0,1[$ for any $a < a_2 = 1 + \sqrt{6}$. More generally, there exists an increasing sequence $(a_k)_k$ of bifurcation values such that for $a_k < a < a_{k+1}$, the asymptotic regime is a cycle of period 2^k, which destabilizes in a_{k+1} through a pitchfork bifurcation of $f_a^{2^k}$. This sequence converges to $a_\infty \approx 3.5699$ according to the scaling law $a_\infty - a_k \sim \delta^{-k}$ with a universal rate $\delta \approx 4.6692$ [6]. The discrete evolution (1) is actually a generic example exhibiting this so-called period-doubling scenario towards chaos, i.e. a normal form to which any one-parameter family experiencing such a scenario is conjugated [7]. In $a = a_\infty$, a chaotic behavior arises, reflecting for $a > a_\infty$ in a positive Lyapounov exponent (sensitivity to initial conditions) and mixing property (dynamic decorrelation of phase space regions). Chaotic regions in the a-space then intermingle in a highly complicated fashion (but now understood [7]) with non chaotic regions where stable odd cycles rule the asymptotic dynamics.

The conclusion, now acknowledged but striking at the time of publication of Ref. [2] and anyhow remarkable, is that a large variety of chaotic behaviors can be generated by a one-dimensional discrete evolution, with a seemingly harmless nonlinearity (smooth and simply quadratic). It showed that nonlinearities are never harmless when supplemented with a folding dynamics, here coming from the bell shape of the evolution map.

II. CONTINUOUS-TIME COUNTERPART: A TRIVIAL DYNAMICS

As it is impossible to give an analytical solution of (1), i.e. x_n as an explicit function of n and x_0, and because we are interested in the asymptotic solution $n \to \infty$ (which gives a vanishing relative duration to the unit step $n \to n+1$) it is appealing to deal with the corresponding continuous problem [9], which is straightforwardly solvable. To derive a continuous counterpart of (1), one subtracts x_n to both sides of equation (1) and identifies $x_{n+1} - x_n$ with the differential of a continuous function of time $y(t)$, which leads:

$$\frac{dy}{dt} = f_a(y) - y = y[a(1-y) - 1], \tag{2}$$

whose analytical solution is easily obtained :

$$y(t) = \frac{(a-1)y_0}{ay_0 + [a(1-y_0) - 1]e^{-(a-1)t}}. \tag{3}$$

This solution is obviously regular with respect to $t \geq 0$ for any value of $a > 1$ and, not surprisingly, tends to x_a^* when $t \to \infty$. In contrast with this plain behavior, qualitatively insensitive to the value of $a > 1$, any attempt to solve (2) by discretization with a time step $h = 1$ will lead to the logistic evolution (1) with its full richness of solutions as a is varied. On the other hand one expects that, for h small enough, one should approach the true solution (3). How is it possible ? We have therefore to quantify what means "small enough".

III. INTERPRETATION OF DISCRETIZATION SCHEMES ASSOCIATED WITH THE LOGISTIC EQUATION

Let us thus recall the behavior of the discretization schemes associated with (2)[10]. Our aim is evidently not to get more knowledge about this equation, nor to device an accurate numerical resolution, but rather to understand in this tractable and well-understood situation what is currently done to solve real problems when no straightforward solution is available.

For a given time step h, the discretization scheme writes

$$y(t+h) = y(t) + h\{ay(t)\,[1 - y(t]) - y(t)\} \tag{4}$$

A remarkable feature of the logistic equation is the possibility to rewrite this scheme as

$$Y(t+h) = AY(t)\,(1 - Y(t))\,, \tag{5}$$

with

$$Y(t) = \lambda y(t) \quad \text{where} \quad \lambda = \frac{ah}{1 + a(h-1)} \tag{6}$$

involving the effective control parameter

$$A(a,h) = 1 + h(a-1) \tag{7}$$

provided $y_0 \in [0, 1/\lambda]$ (note that $\lambda < 1$ if $h < 1$). Obviously, the same phenomenology as for evolution (1) will be observed. For instance, the inequality $A < 3$, required to obtain the convergence of (5) to the nontrivial fixed point $Y_A^* = 1 - 1/A$, means

$$h < h_c(a) = \frac{a_1 - 1}{a - 1} = \frac{2}{a-1} \tag{8}$$

Extending the reasoning to the subsequent bifurcations, one would observe a period-doubling scenario when the discretization step h increases, namely at values $(h_k)_k$ with $A(a, h_k) = a_k$, i.e.

$$h_k = \frac{a_k - 1}{a - 1} \tag{9}$$

Chaos arises for $h > h_\infty(a) = (a_\infty - 1)/(a - 1)$. The bifurcation diagram as a function of h, at fixed a, would then be similar to the standard bifurcation diagram in a-space, up to a rescaling of the attracting sets by a factor of $\lambda(a, h)$, and a translation and rescaling of the bifurcation values ($a_k = 1 + (a - 1)h_k$). In particular, it is interesting to note that the sequence $(h_k)_k$ follows the same universal scaling law $h_\infty - h_k \sim \delta^{-k}$ or more precisely:

$$\frac{h_{i+1} - h_i}{h_{i+2} - h_{i+1}} \longrightarrow \delta \quad \text{when} \quad i \to \infty \quad \text{with} \quad \delta \approx 4.4669 \tag{10}$$

For illustration let us consider the case $a = 3.1$ (Figures 1, 2 and 3). The critical value of h is $h_c = (a_1)/(a - 1) = 2/2.1 \simeq 0.9524$. For $h > h_c$, one sees a 2-cycle, namely oscillations of the solution between the two (stable) fixed points of $f_A[f_A(Y)]$. The onset of the chaos is for $h = h_\infty = (a_\infty - 1)/(a - 1) = 2.5699/2.1 = 1.22376$.

IV. DISCUSSION: AN INTERPLAY BETWEEN TWO CHARACTERISTIC TIMES

This simple study illustrates that the passage from continuous to discrete nonlinear equation is not insignificant: destabilization of the continuous time evolution, leading to cycles and even a spurious chaotic behavior, follows from an improper choice of the step of the discretization [11] or conversely an actual chaotic behavior can be suppressed by replacing a discrete model by its limiting continuous counterpart.

Nevertheless, the passage from equation (1) to (5) by a simple scaling is exact only in the case of the quadratic family. We shall now investigate what remains in more general situations. Let f be a map, generating a discrete dynamical system $x_{n+1} = f(x_n)$ and having a stable fixed point x^* (i.e. $f(x^*) = x^*$ and $|f'(x^*)| < 1$). The naive continuous counterpart writes $dy/dt = f(y) - y$. Linear stability analysis shows that x^* is still a (at least locally) stable fixed point of the continuous dynamics since the linear growth rate of perturbations is negative: $f'(x^*) - 1 < 0$.

We might then consider the discrete scheme $z_{n+1} = z_n + h[f(z_n) - z_n]$ for various values of the time step h. It is straightforward to show that this discretization scheme destabilizes for $h > h_c$ where

$$h_c = 2/[1 - f'(x^*)] \tag{11}$$

Indeed, the linear stability of x^* breaks down when the modulus $|1 + h(f'(x^*) - 1)|$ overwhelms 1, which occurs for $1 + h(f'(x^*) - 1) = -1$. This relation yields the above value of h_c and shows that the discrete scheme exhibits a period-doubling (pitchfork) bifurcation in $h = h_c$ (the additional generic condition for this bifurcation stated in Section 1 being also fulfilled, as can be directly checked).

The additional feature observed when the map f_a depends on a control parameter a and exhibits a period-doubling in a_1 is that $h_c(a)$ crosses $h = 1$ in $a = a_1$: for $a > a_1$, $f'_a(x^*_a) < -1$ and x^*_a is instable with respect to the initial discrete dynamics ($h = 1$) but is still a stable fixed point of the continuous dynamics, showing the inadequacy of the limiting continuous model $dy/dt = f_a(y) - y$ to capture the behavior of the discrete one $x_{n+1} = f_a(x_n)$. It is to note that $h_c(a)$ decreases if a increases: the more stable is the fixed point (i.e. the larger $|f'_a(x^*_a) - 1|$ with $f'_a(x^*_a) - 1 < 0$), the smaller is the time-step range of validity of the discretization scheme (in a sense, the less stable is the discretization scheme).

The qualitative differences, explicitly described in the previous sections, between the continuous-time and discrete-time versions of the logistic equation (and above in a more general framework) are not really surprising: a general claim assesses that a continuous-time dynamics requires a phase space of dimension at least 3 to develop a chaotic behavior [13]. In dimension 1 or 2, continuous trajectories behave as boundaries each for each other (trajectories of an autonomous continuous dynamic system cannot cross each other), which obviously prevents from chaos (and even nontrivial periodic solutions in dimension 1). But whereas it is thus straightforward to foresee the loss of chaotic and even periodic behavior when turning to the limiting continuous dynamics, is it possible to understand on physical grounds the existence of a critical value h_c for the discretization time step h? The explanation lies in the comparison of the intrinsic time scale(s) of the dynamics with the chosen "time unit" h.

The characteristic time of a continuous evolution, still denoted $dy/dt = f(y) - y$ to avoid proliferation of new notations, can be estimated as $\tau \sim 1/[1 - f'(x^*)]$. Indeed, a mere linearization of (2) around the fixed point x^* leads to:

$$\frac{d}{dt}[x(t) - x^*] = [f'(x^*) - 1](x - x^*) \tag{12}$$

hence the value of τ. Destabilization of the discretization scheme occurs when $h > h_c = 2\tau$. The stepwise updating, after each time step h, of the evolution law is too rough to properly control the discrete evolution and force it to follow closely all the relevant variations of the continuous

trajectory. This is reminiscent of the Nyquist theorem [12] for a periodic continuous evolution: the observation time step should be smaller than half the smallest period (or characteristic time) to properly sampling the continuous trajectory.

It is to note that τ or equivalently the critical value $h_c = 2\tau$ of the time step are intrinsic features of the dynamics, in the sense that they are invariant through conjugacy: for any diffeomorphism ϕ, $f(y)$ and $\phi^{-1} \circ f \circ \phi(y)$ (providing an equivalent modeling of the discrete model associated with f) or $f(y)$ and $y + \phi^{-1}[f \circ \phi(y) - \phi(y)]$ (providing an equivalent modeling of the continuous model associated with $f(y) - y$) will have the same critical value h_c and the same characteristic time τ.

Let us carry further the comparison between the continuous evolution and its discretization, in order to understand the emergence of oscillations for $h > 2\tau$. The general continuous equation $dy/dt = f(y) - y$ operates a fine tuning of the evolution rate dy/dt that is obviously not achieved by updating $f(y) - y$ at times $t_n = nh$. We have shown here that, near a stable fixed point, the resulting discrepancies lead to a bifurcation in the asymptotic dynamics, when h overwhelms the characteristic time of the evolution. To take a familiar example of such oscillations arising from a mismatch between two characteristic times, let us consider an heating/cooling device, able to measure the difference between the instantaneous room temperature and a prescribed one, and to monitor the appropriate energy supply or extraction, to compensate the measured difference. If the time h necessary for the device to actually deliver the required energy is longer that the characteristic time of temperature variations in the environment, the device will not balance the external temperature variations but rather, its ill-phased response will superimpose and the room temperature will suffer large oscillations. More generally, any ill-tuned homeostatic device, responding with a large lag h, will produce oscillations, and the result of Section 3 is the mathematical translation of this ubiquitous phenomenon.

V. CONCLUSION

In conclusion, we have presented an example showing explicitly the link between the validity of the discretization scheme with the dynamical (in)stability of the associated map for a unit step-size. Convertely, it enlights the specificity of the discrete dynamics, that cannot in general be understood, even qualitatively, from the behaviour of its continuous counterpart. In two or more dimensions, an additional problem arises: : the recursion relation is no more unique [14]. The caveats illustrated in this paper are all the more relevant.

[1] P.F. Verhulst, Notice sur la loi que la population suit dans son accroissement, *Corresp. Math. et Phys.* **10** (1838) 13–21.

[2] R. M. May, Simple mathematical models with very complicated dynamics, Nature, **261** (1976) 459–467. Witout lowering the historical importance and repercussions of this paper, it is to note that more is known today on the asymptotic behavior in the region $a > a_c$, which leads to modify May's claim that all trajectotires are periodic but with period so large that the dynamics resembles chaos. In fact, in the region $a > a_c$, windows of a-values where the dynamics is actually periodic intermingle in a very intricate way with windows where a definite chaotic behavior (sensitivity to initial conditions, strictly positive Lyapounov exponent) is observed; see for instance ref.[7] below.

[3] H.O. Peitgen, H. Jürgens and D. Saupe, *Chaos and fractals*, Springer, Berlin (1992).

[4] H.J. Korsch and H.J. Jodl, *Chaos: a program collection for the PC*, Springer, New York (1998).

[5] G. Iooss and D.D. Joseph, *Elementary stability and bifurcation theory*, Springer, Berlin (1981).

[6] M. Feigenbaum, Quantitative universality for a class of nonlinear transformations, *J. Stat. Phys.* **19** (1978) 25-52. P. Coullet and C. Tresser, Itération d'endomorphismes et groupe de renormalisation, *CR Acad. Sc. Paris* **287A** (1978), 577-580.

[7] P. Collet and J.P. Eckmann, *Iterated maps of the interval as dynamical systems*, Birkhaüser, Boston (1981).

[8] Except for $a = 4$, where $x_n = \sin^2(2\pi\theta_n)$ with $\theta_n = 2\theta_{n-1} = 2^n\theta_0$ if $x_0 = \sin^2(2\pi\theta_0)$. This equivalence with the angle-doubling $\theta_{n+1} = 2\theta_n$ (modulo 1) allows to prove that one gets a fully chaotic behavior for $a = 4$ (the location of x_n below 1/2, coded 0, or above 1/2, coded 1, generates a binary sequence that is statistically equivalent to the outcome of a game of heads-and-tails).

[9] J. Hubbard and B. West, *Differential equations, a dynamical systems approach*, Springer, New York (1991).

[10] R.L. Borrelli and C.S. Coleman Differential equations: A modeling perspective, Wiley, New York (1998).

[11] M. Yamaguti and H. Matano, Euler's finite difference scheme and chaos, Proc. Japan Acad. Series A, **55** (1979) 78–80.

[12] C. Shannon, Communication in the presence of noise, *Proceedings of the IRE* **37** (1949) 10-21. H. Nyquist, Certain topics in telegraph transmission theory, *AIEE Trans.* **47** (1928) 617-644.

[13] R.C. Hilborn, *Chaos and nonlinear dynamics*, Oxford University Press (1994).

[14] H. Krivine, A. Lesne and J. Treiner, Discretization schemes for conservative systems dynamical systems: Two case studies (in preparation).

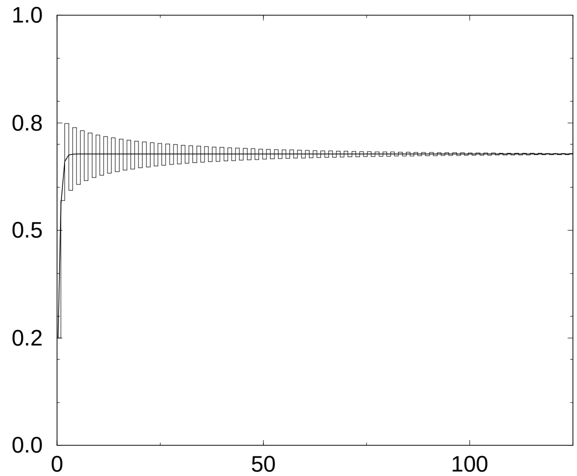

FIG. 1: *Discretization of the logistic equation (2) with* $a = 3.1$*, using a time step* $h < h_c$ *(here* $h = 0.94$ *whereas* $h_c \equiv 2/(a-1) = 20/21 \approx 0.95$*), see text, Section 3. Bold line: exact (continuous-time) solution of (2). Stair step* $\frac{1}{\lambda} f_A(nh)$*.* $x_a^* = 1 - 1/a$*.*

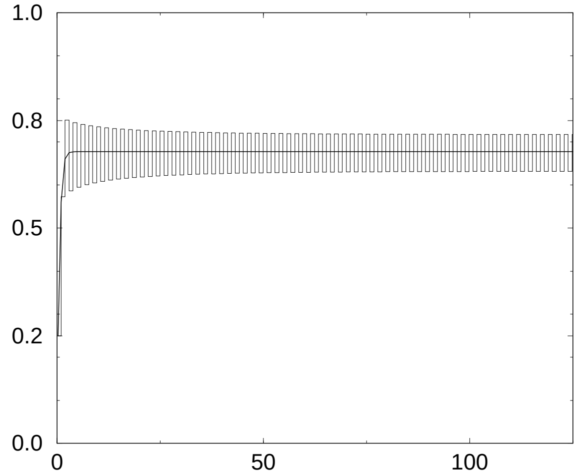

FIG. 2: *Same as Fig. 1 but with* $h = 0.96 > h_c$.

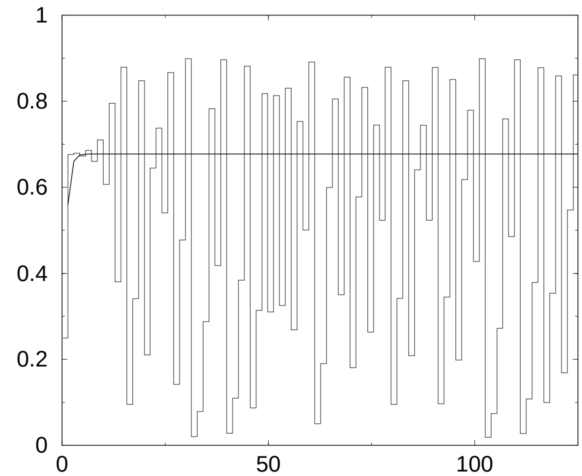

FIG. 3: *Discretization of the logistic equation (2) with* $a = 3.1$*, using a time step* $h = 3/(a-1) = 1.424$ *corresponding to the fully chaotic case* $A = 4$*, see text, Section 3, and [8].*